Class 16, given on Feb 8, 2010, for Math 13, Winter 2010

1. Line integrals of vector fields: definition

Now that we've defined vector fields, look at a few examples of vector fields, and also defined the special class of conservative vector fields, let's look at how to evaluate line integrals of vector fields.

The definition of the line integral of a vector field $\mathbf{F}$ along a curve C is similar to the definition of the line integral of a scalar function along C. This definition should allow us to calculate the work a force (possibly varying with position) exerts on a particle as that particle travels in a curve C.

Suppose C is a curve in $\mathbb{R}^2$, parameterized by the vector-valued function $\mathbf{r}(t), a \leq t \leq b$. If $\mathbf{F}$ is a vector field defined on C, then the *line integral of* $\mathbf{F}$ *along* C is given by the formula

$$\int_C \mathbf{F}\, d\mathbf{r} = \int_a^b \mathbf{F}(x(t), y(t)) \cdot \mathbf{r}'(t)\, dt.$$

Notice the similarity of this formula to the analogous formula for the line integral of a scalar function. If we think of $\mathbf{r}(t)$ as describing the position of a particle, recall that $\mathbf{r}'(t)$ describes the instantaneous velocity of that particle. Therefore, if $\mathbf{F}$ is interpreted as a force, then $\mathbf{F}(x(t), y(t)) \cdot \mathbf{r}'(t)$ describes the work per second being done on the particle at a given time t, and integrating this expression with respect to t will calculate the total work done over the entire path of the particle.

- Sometimes line integrals of vector fields are written slightly differently than what we mentioned above. If $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ is a description of $\mathbf{F}$ in terms of its components, then we sometimes write the line integral of $\mathbf{F}$ over C as

  $$\int_C \mathbf{F}\, d\mathbf{s} = \int_C P\, dx + Q\, dy.$$

  The interpretation of this notation is that when calculating, say

  $$\int_C P\, dx,$$

  which looks like the line integral of the scalar function P (the only difference being the presence of dx instead of ds as the differential in the integral sign), when we calculate this integral by expressing this as an integral of a function in t, we should replace dx with $x'(t)\, dt$:

  $$\int_C P\, dx = \int_a^b P(x(t), y(t))x'(t)\, dt.$$

  Therefore, with this interpretation of these terms (sometimes called the line integral of P along C with respect to x), we have

  $$\int_C P\, dx + Q\, dy = \int_a^b P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)\, dt = \int_a^b \mathbf{F}'(x(t), y(t)) \cdot \mathbf{r}'(t)\, dt.$$

  Therefore, this different notation for line integrals of vector fields is compatible with the original definition, as it should be.
- When calculating the line integral of a scalar function f along C, we needed to find a parameterization $\mathbf{r}(t)$ for C. However, the parameterization we chose did not matter, as long as $\mathbf{r}(t)$ touched each point of C exactly once. When calculating line integrals of vector fields, the choice of parameterization can affect the value of the line integral. In particular, given a curve C, there are two choices of directions for C,

called an *orientation* for C. A parameterization corresponds to a given orientation depending on the direction it traverses as t increases; it turns out that the value of a line integral will change sign if the orientation describing C is reversed. However, if two parameterizations have the same orientation, the value of the line integral will be the same using either of the parameterizations.

- We initially define line integrals over curves C which can be described by a differentiable function $\mathbf{r}(t)$. We extend the definition to curves which are piecewise differentiable – that is, curves which are everywhere differentiable except at a finite number of points – by adding up the value of the line integrals over each of the individual pieces.

Let us look at some examples illustrating the calculation of line integrals over vector fields, as well as some of the remarks above.

Examples.

- Let $\mathbf{F} = \langle y, 2x\rangle$, and let C be the curve which is the line segment from $(0,0)$ to $(1,1)$. Suppose C is given by parameterization $\mathbf{r}(t) = \langle t, t\rangle, 0 \leq t \leq 1$. Calculate the line integral of $\mathbf{F}$ on C using this parameterization.

  We first remark that we can write this line integral in either of the forms
$$\int_C \mathbf{F} d\mathbf{r} = \int_C y\,dx + 2x\,dy.$$
  We have $\mathbf{r}'(t) = \langle 1, 1\rangle$ and $\mathbf{F}(x(t), y(t)) = \langle t, 2t\rangle$. Therefore, this line integral is equal to
$$\int_0^1 \langle t, 2t\rangle \cdot \langle 1, 1\rangle\,dt = \int_0^1 3t\,dt = \frac{3}{2}.$$
- Let $\mathbf{F} = \langle y, 2x\rangle$, and let C be the curve which is the line segment from $(0,0)$ to $(1,1)$. Suppose C is given by the parameterization $\mathbf{r}(t) = \langle t^2, t^2\rangle, 0 \leq t \leq 1$. Calculate the line integral of $\mathbf{F}$ along C using this parameterization.

  This time, we have $\mathbf{r}'(t) = \langle 2t, 2t\rangle$. Therefore, the line integral in question is equal to
$$\int_0^1 \langle t^2, 2t^2\rangle \cdot \langle 2t, 2t\rangle\,dt = \int_0^1 2t^3 + 4t^3\,dt = \frac{6}{4} = \frac{3}{2}.$$
  This example illustrates the fact that, even though this example and the previous example have different parameterizations for C, the resulting value of the line integral is the same, since both parameterizations describe the same orientation for C.
- Let $\mathbf{F} = \langle y, 2x\rangle$, and let C be the curve which is the line segment from $(0,0)$ to $(1,1)$. (This is the same setup as the previous two examples!) Suppose C is given by parameterization $\mathbf{r}(t) = \langle 1-t, 1-t\rangle, 0 \leq t \leq 1$. Calculate the line integral of $\mathbf{F}$ on C using this parameterization.

  This example differs from the previous two in that while $\mathbf{F}, C$ are the same, the parameterization for C is a different orientation now. We have $\mathbf{r}'(t) = \langle -1, -1\rangle$, so the line integral in question equals
$$\int_0^1 \langle 1-t, 2(1-t)\rangle \cdot \langle -1, -1\rangle\,dt = \int_0^1 -3(1-t)\,dt = \frac{3(1-t)^2}{2}\Big|_0^1 = \frac{-3}{2}.$$
  The answer is the negative of the answer from the previous two examples, exactly as expected given our remarks earlier.

- Let $\mathbf{F} = \langle y, 2x \rangle$, and let C be the curve given by parameterization $\mathbf{r}(t) = \langle t, t^2 \rangle, 0 \leq t \leq 1$. Calculate the line integral of $\mathbf{F}$ along C using this parameterization.

  This example differs from the previous three in that C is no longer a line segment, but a piece of a parabola. However, this C has the same endpoints as the C used in the previous examples.

  We have $\mathbf{r}'(t) = \langle 1, 2t \rangle$. Therefore, the line integral of $\mathbf{F}$ along C is

$$\int_0^1 \langle t^2, 2t \rangle \cdot \langle 1, 2t \rangle \, dt = \int_0^1 5t^2 \, dt = \frac{5}{3}.$$

Notice that this value differs from the previous three examples. This should not be too surprising since there is no a priori reason to believe that the value of a line integral should depend only on the endpoints of C. However, there is a special class of vector fields for which the values of line integrals along various paths C do not depend on C, and only depend on the endpoints of C.

2. The Fundamental Theorem of Calculus for line integrals

Suppose that $\mathbf{F}$ is a conservative vector field; that is, $\mathbf{F} = \nabla f$ for some scalar function f. We initially assume that this relation holds true for all (x, y) in $\mathbb{R}^2$, although later we will see how we can relax this condition. Then the following theorem holds true:

Theorem. (The Fundamental Theorem of Calculus for line integrals) Let $\mathbf{F} = \nabla f$ be a continuous vector field for some differentiable scalar function $f(x, y)$ on $\mathbb{R}^2$. Let C be a curve parameterized by $\mathbf{r}(t), a \leq t \leq b$. Then

$$\int_C \mathbf{F} \, d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Remarks.

- Notice the strong resemblance of this theorem to the usual FTC. Some sort of definite integral is equal to the value of a function at one point subtracted from the value of a function at another point. The role of an antiderivative for $\mathbf{F}$ is played by the scalar function f, and the role of the endpoints of the interval $[a, b]$ are played by the endpoints of C, $\mathbf{r}(a), \mathbf{r}(b)$.
- One consequence of this theorem is that if $\mathbf{F}$ is conservative on $\mathbb{R}^2$, then the value of a line integral of $\mathbf{F}$ along C does not actually depend on the path taken by C, but only on the endpoints of C. That is, a phenomenon like that which we observed in the previous set of examples can never take place for a conservative vector field. We say that a line integral of $\mathbf{F}$ over C is *independent of path* if its values only depend on the endpoints of C. In this terminology, the line integral of a conservative vector field is independent of path for all curves C.
- A special case of the above case occurs when we calculate the line integral of $\mathbf{F}$ along a curve C whose starting and endpoints are the same. If C is such a curve, we call C a closed curve, and the line integral of a conservative vector field along a closed curve C is equal to

$$\int_C \mathbf{F} \, d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = 0,$$

  since $\mathbf{r}(b) = \mathbf{r}(a)$. Therefore, the line integral of a conservative vector field along a closed path is always equal to 0, regardless of the shape of the closed path.

- The resemblance of this theorem to the usual FTC is not accidental. As a matter of fact, the proof of this theorem (which we do not cover here) uses the usual FTC along with the chain rule applied to $\mathbf{F}(x(t), y(t))$.

One nice application of the above theorem is that it becomes possible to calculate the line integral of conservative vector fields even when the path C is very strange or complicated.

Example. Let $\mathbf{F} = \langle ye^x, e^x \rangle$, and let C be given by parameterization $\mathbf{r}(t) = \langle t^{17}, \cos^3 \pi t \rangle$, $0 \leq t \leq 1$. Calculate the line integral of $\mathbf{F}$ along C.

If you try to calculate this line integral directly, you will end up with a complete mess of an expression in the function t. The much faster way to calculate this line integral is to recognize $\mathbf{F}$ as the gradient of $f(x, y) = ye^x$. One way of finding this function is to pretend that there is an $f(x, y)$ with $\nabla f = \mathbf{F}$. If this is the case, then $f_x = ye^x, f_y = e^x$. In particular, the latter expression forces $f(x, y)$ to have the form $f(x, y) = ye^x + g(y)$, for some function $g(y)$ solely of y, while the former expression requires $f(x, y) = ye^x + h(x)$, for some function $h(x)$ solely of x. Since these two expressions must be equal, we have $f(x, y) = ye^x$, since $g(y) = h(x)$ is only possible if both expressions are constant. (We could have chosen $f(x, y) = ye^x + C$ for any constant C, but this does not affect the final answers.)

Since $\mathbf{F}$ is the gradient of a function differentiable on all of $\mathbb{R}^2$, we have

$$\int_C \mathbf{F}\, d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)).$$

Since $\mathbf{r}(1) = \langle 1^{17}, \cos^3 \pi \rangle = \langle 1, -1 \rangle, \mathbf{r}(0) = \langle 0^{17}, \cos^3 0 \rangle = \langle 0, 1 \rangle$, this expression is equal to

$$f(1, -1) - f(0, 1) = -e^1 - 1 = -(e + 1).$$

This example suggests one natural, very important question: given a vector field $\mathbf{F}$, is there a quick way of determining whether it is conservative? In this example we were able to take ‘partial integrals’ to find a potential function $f(x, y)$ for $\mathbf{F}$. However, there may be situations where we cannot take partial integrals.

Another question we still want to answer is how to generalize the fundamental theorem for line integrals to vector fields which may not be defined on all of $\mathbb{R}^2$. The most natural example of such a vector field is the gravitational/electric field generated by a particle. This field is not defined at the origin, but is conservative everywhere else, so we want to know if the fundamental theorem holds true for this vector field as well.